Open-Source Toolbox for Semi-Analytical Hydrodynamic Coefficients via the Matched Eigenfunction Expansion Method

Rebecca McCabe, Kapil Khanal, and Maha Haji

Symbiotic Engineering and Analysis Laboratory
Cornell University

Agenda

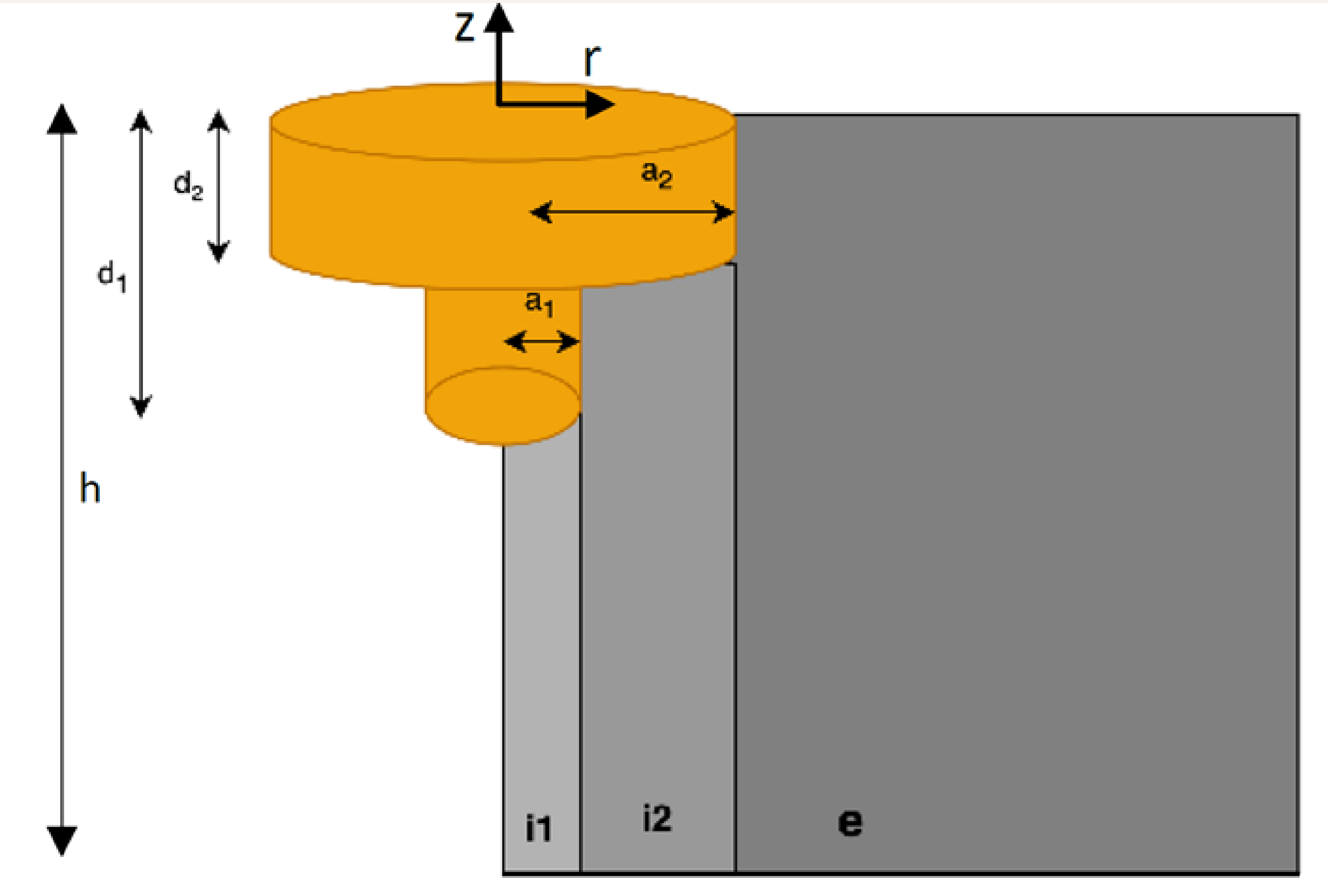
- Motivation
- Method
    - Problem setup
    - Continuity across fluid boundaries
    - Complex linear system
- Numerical notes
    - Validation
    - Convergence
    - Runtime

Motivation

- Semi-analytic (mesh-free) solutions for simple geometries
- Faster than boundary element method, no irregular frequencies
- Easily extendable to yield derivatives
- Use cases:
  - Design optimization, sensitivity analysis, control co-design
  - Benchmarking for numerical methods
- Decades-old method, but code not available for broad use

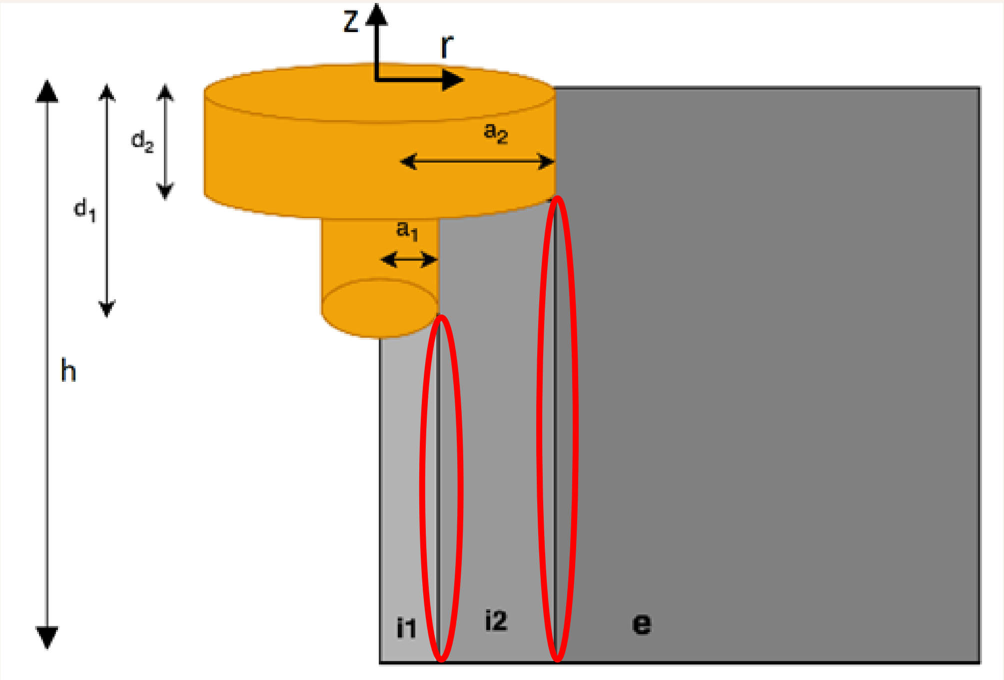
Problem Setup

- Geometry
  - Two axisymmetric cylinders
  - Oscillating independently
- Separable Laplace PDE
  - Linear potential flow
  - Boundary conditions
  - Separate fluid regions
- Potential is the infinite sum of a linear combination of unknown eigen-coefficients and known orthogonal eigenfunctions



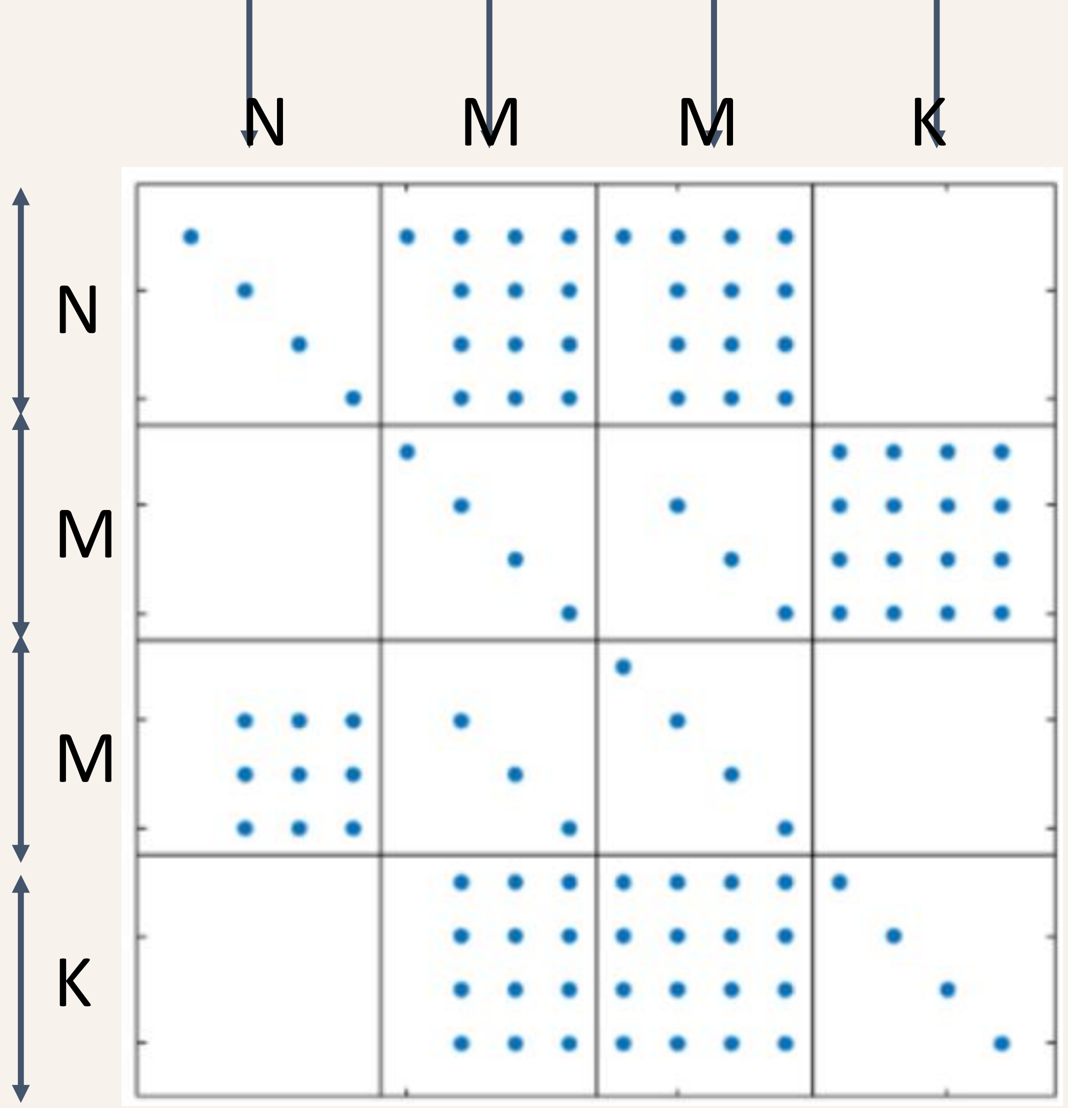
Continuity Across Fluid Boundaries

- 4 matching equations
  - Match potential at $r=a_1$
  - Match potential at $r=a_2$
  - Match radial velocity at $r=a_1$
  - Match radial velocity at $r=a_2$
- Truncate infinite summations to N, M, K terms in regions i1, i2, e respectively
- 4 equations become N+2M+K equations by using eigenfunction orthogonality property
- Results in complex linear system for the unknown eigen-coefficients



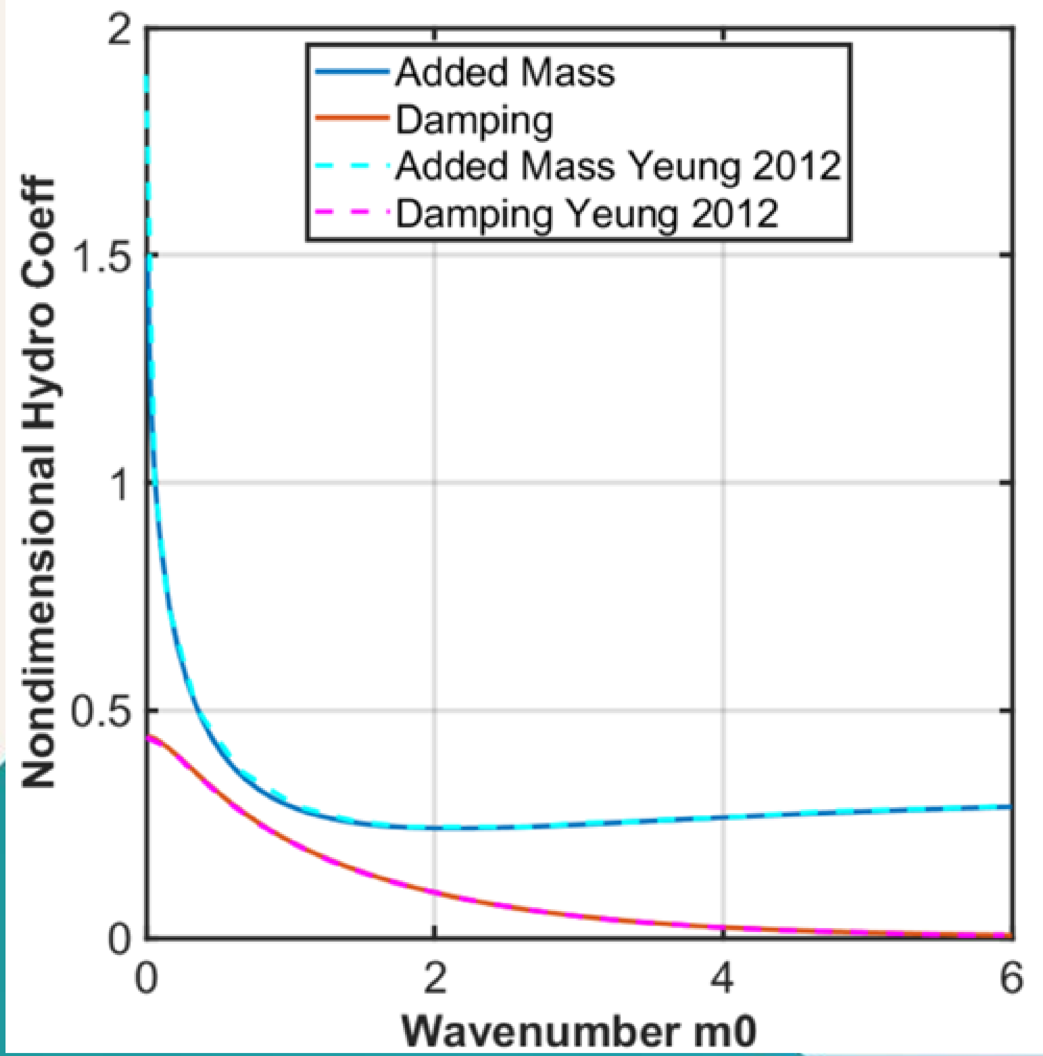
Complex Linear System

- Ax=b
- A is complex sparse block matrix
  - Elements computed with Bessel functions and trigonometric integrals
  - High condition number (>1e14) if N,M high
- b is real vector from particular solution
- Linear solve for eigen-coefficients (x)
- Hydro coefficients are a linear combination of eigen-coefficients with geometric ratios

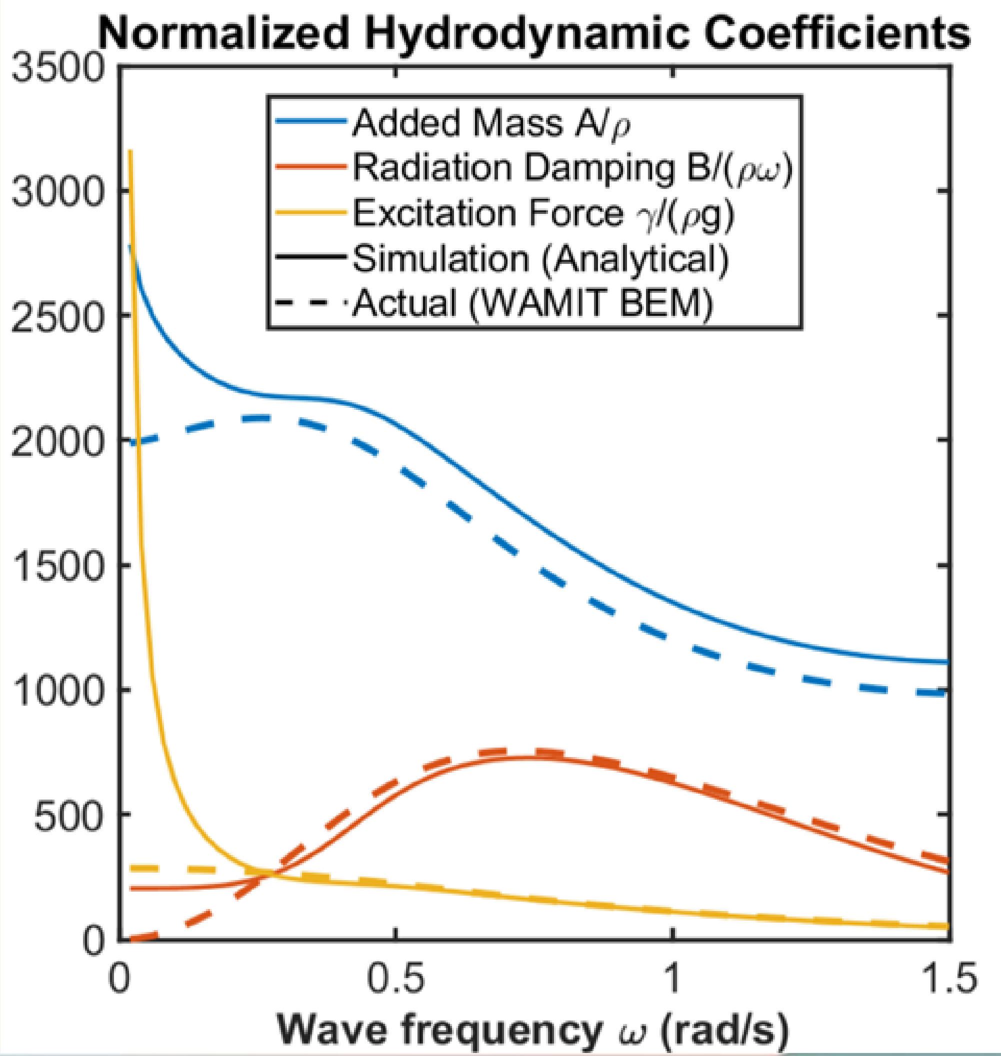


A-matrix sparsity for N=M=K=4

Validation



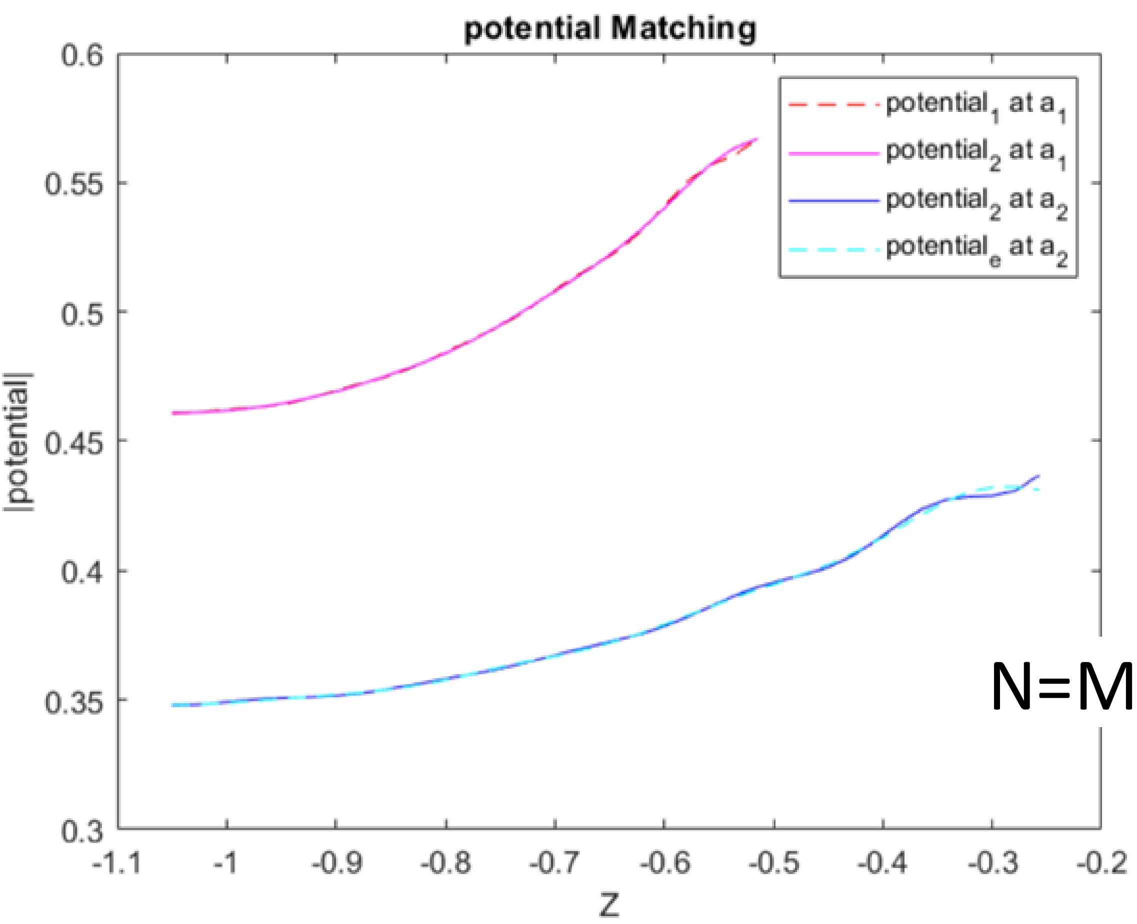
Added Mass
Damping
Added Mass Yeung 2012
Damping Yeung 2012
Nondimensional Hydro Coeff
Wavenumber m0
0
0.5
1
1.5
2
0
2
4
6



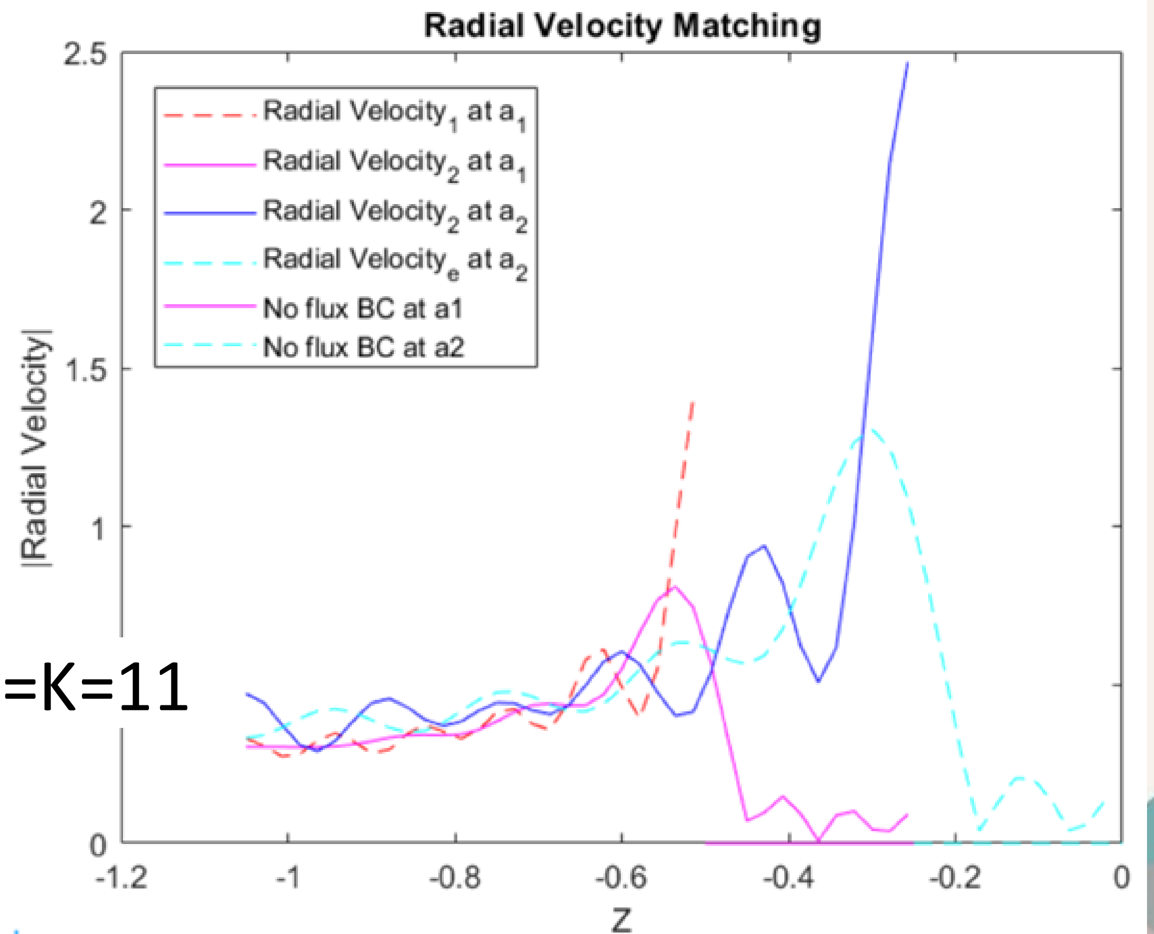
Normalized Hydrodynamic Coefficients
Added Mass A/ρ
Radiation Damping B/(ρω)
Excitation Force γ/(ρg)
Simulation (Analytical)
Actual (WAMIT BEM)
0
500
1000
1500
2000
2500
3000
3500
0
0.5
1
1.5
Wave frequency ω (rad/s)

Convergence at Boundaries

Potential converges faster than velocity

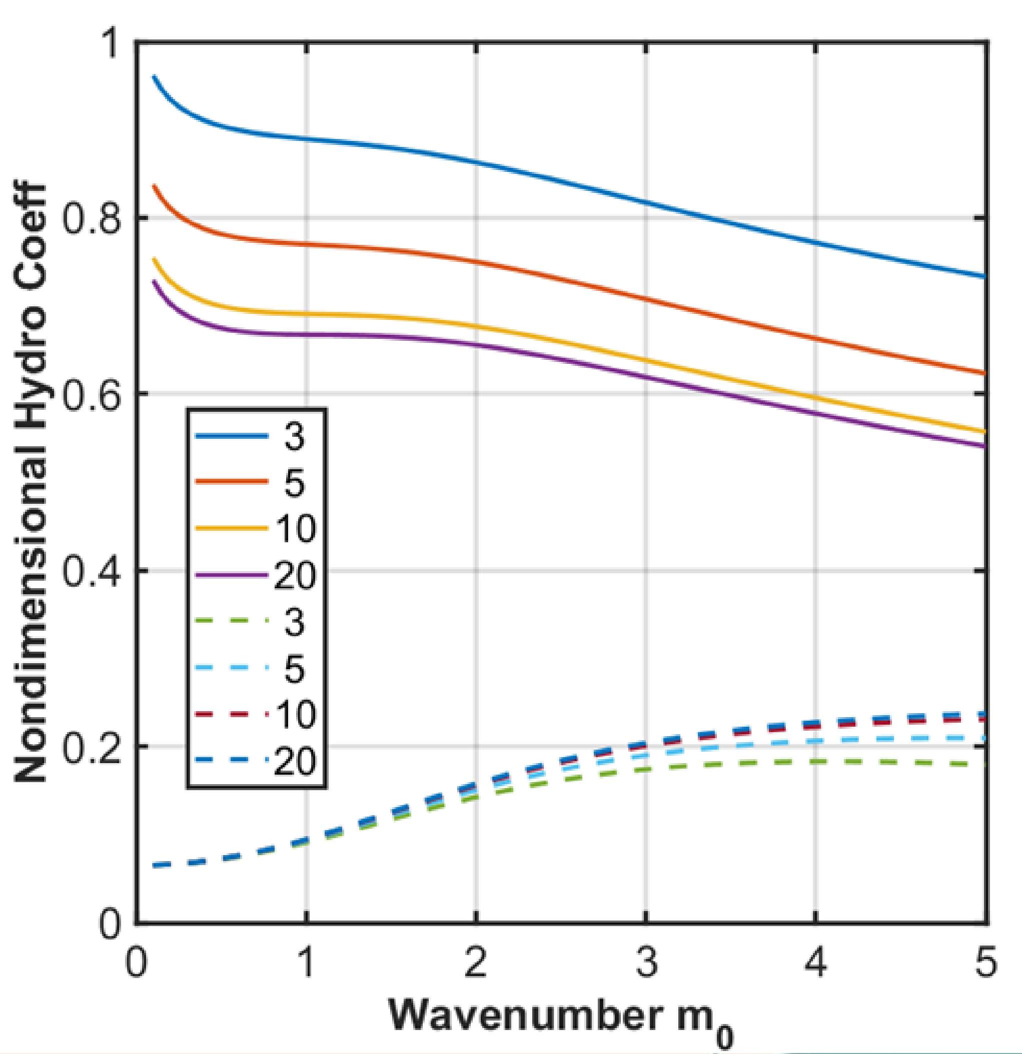


N=M=K=11



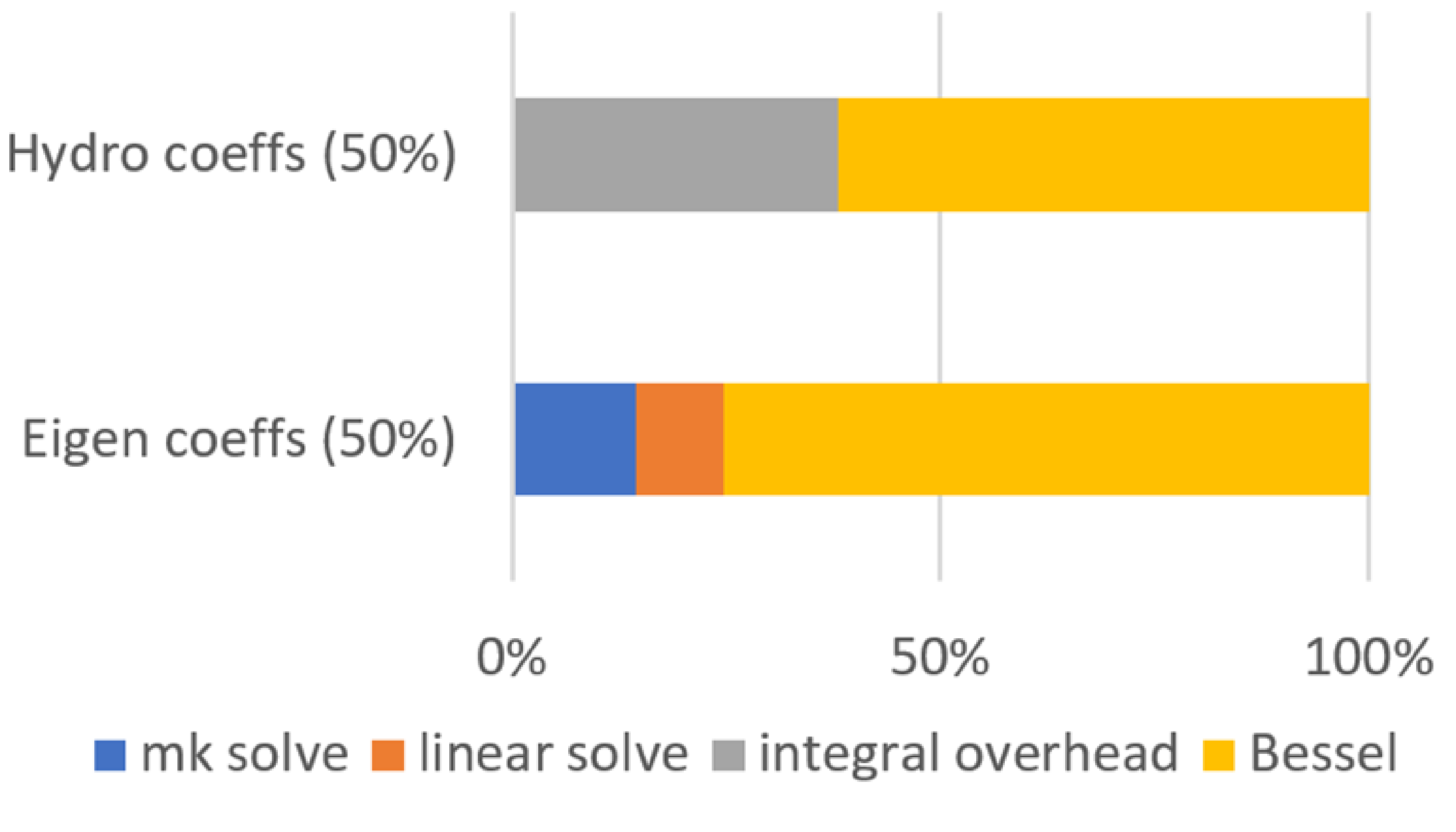
Convergence of Hydro Coefficients

- Benchmark geometry converges to within 0.25% with only N=M=K=4
- RM3 geometry requires N=M=K>11
  - Damping converges better at low frequencies
  - Added mass converges similarly across frequencies



Runtime and Computational Cost Scaling

- Most time spent on Bessel function evaluations
- 10x faster than Capytaine for 1% convergence (31 vs 323 ms)
- Possible speedup: change numeric integration to analytic



Future Work

- Extend to other modes of motion
- More fluid domains: model arbitrary axisymmetric geometries
- Study of truncation terms and their convergence for each domain
- Compute gradients with respect to geometry and wave parameters
- Couple with optimizer for design optimization

Code Accessibility

- MATLAB code (open source with MIT license): https://github.com/symbiotic-engineering/MDOcean/blob/main/mdocean/simulation/modules/MEEM/run_MEEM.m
- Python code: coming soon, intended as primary

Acknowledgements

We thank Prof. R. W. Yeung and Seung-Yoon Han for discussions on the theory and computation of this method.

This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE–2139899. Any opinion, findings, and conclusions or recommendations expressed in this material are those of the authors(s) and do not necessarily reflect the views of the National Science Foundation.

Works Cited

[1] I. K. Chatjigeorgiou, *Analytical Methods in Marine Hydrodynamics*. Cambridge: Cambridge University Press, 2018. doi: 10.1017/9781316838983.

[2] F. P. Chau and R. W. Yeung, “Inertia and Damping of Heaving Compound Cylinders,” presented at the 25th International Workshop on Water Waves and Floating Bodies, Harbin, China, Jan. 2010. Accessed: Sep. 27, 2023. [Online]. Available: https://www.academia.edu/73219479/Inertia_and_Damping_of_Heaving_Compound_Cylinders_Fun

[3] F. P. Chau and R. W. Yeung, “Inertia, Damping, and Wave Excitation of Heaving Coaxial Cylinders,” presented at the ASME 2012 31st International Conference on Ocean, Offshore and Arctic Engineering, American Society of Mechanical Engineers Digital Collection, Aug. 2013, pp. 803–813. doi: 10.1115/OMAE2012-83987.

[4] R. W. Yeung, “Added mass and damping of a vertical cylinder in finite-depth waters,” *Appl. Ocean Res.*, vol. 3, no. 3, pp. 119–133, Jul. 1981, doi: 10.1016/0141-1187(81)90101-2.

[5] D. Son, V. Belissen, and R. W. Yeung, “Performance validation and optimization of a dual coaxial-cylinder ocean-wave energy extractor,” *Renew. Energy*, vol. 92, pp. 192–201, Jul. 2016, doi: 10.1016/j.renene.2016.01.032.

[6] K. Kokkinowrachos, S. Mavrakos, and S. Asorakos, “Behaviour of vertical bodies of revolution in waves,” *Ocean Eng.*, vol. 13, no. 6, pp. 505–538, Jan. 1986, doi: 10.1016/0029-8018(86)90037-5.

Appendix: Equations for Potential & Eigenfunctions

Region	i1	i2	e
Homog. potential $\phi_h(r,z)$	$\sum_{n=0}^{\infty} C_{1n}^{i1} R_{1n}^{i1}(r) Z_n^{i1}(z)$	$\sum_{m=0}^{\infty} \left(C_{1m}^{i2} R_{1m}^{i2}(r) + C_{2m}^{i2} R_{2m}^{i2}(r) \right) Z_m^{i2}(z)$	$\sum_{k=0}^{\infty} B_k^e \Lambda_k(r) Z_k^e(z)$
Partic. potential $\phi_p(r,z)$	$\begin{cases} \frac{1}{2(h-d_1)}\left[(z+h)^2 - \frac{r^2}{2}\right], & 1M \\ 0, & 1S \end{cases}$	$\begin{cases} \frac{1}{2(h-d_2)}\left[(z+h)^2 - \frac{r^2}{2}\right], & 2M \\ 0, & 2S \end{cases}$	0
Radial eigen-functions	$R_{1n}^{i1}(r) = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{I_0(\lambda_n^{i1} r)}{I_0(\lambda_n^{i1} a_2)}, & n \geq 1 \end{cases}$	$R_{1m}^{i2}(r) = \begin{cases} \frac{1}{2}, & m = 0 \\ \frac{I_0(\lambda_m^{i2} r)}{I_0(\lambda_m^{i2} a_2)}, & m \geq 1 \end{cases}$ $R_{2m}^{i2}(r) = \begin{cases} \frac{1}{2}\ln\left(\frac{r}{a_2}\right), & m = 0 \\ \frac{K_0(\lambda_m^{i2} r)}{K_0(\lambda_m^{i2} a_2)}, & m \geq 1 \end{cases}$	$\Lambda_k(r) = \begin{cases} \frac{H_0^1(m_0 r)}{H_0^1(m_0 a_2)}, & k = 0 \\ \frac{K_0^1(m_k r)}{K_0^1(m_k a_2)}, & k \geq 1 \end{cases}$
Vertical eigen-function	$Z_n^{i1}(z) = \begin{cases} 1, & n = 0 \\ \sqrt{2}\cos(\lambda_n^{i1}(z+h)), & n \geq 1 \end{cases}$	$Z_m^{i2}(z) = \begin{cases} 1, & m = 0 \\ \sqrt{2}\cos(\lambda_m^{i2}(z+h)), & m \geq 1 \end{cases}$	$Z_k^e(z) = \begin{cases} N_0^{1/2}\cosh(m_0(z+h)), & k = 0 \\ N_k^{1/2}\cos(m_k(z+h)), & k \geq 1 \end{cases}$

Appendix: Block Matrix Structure

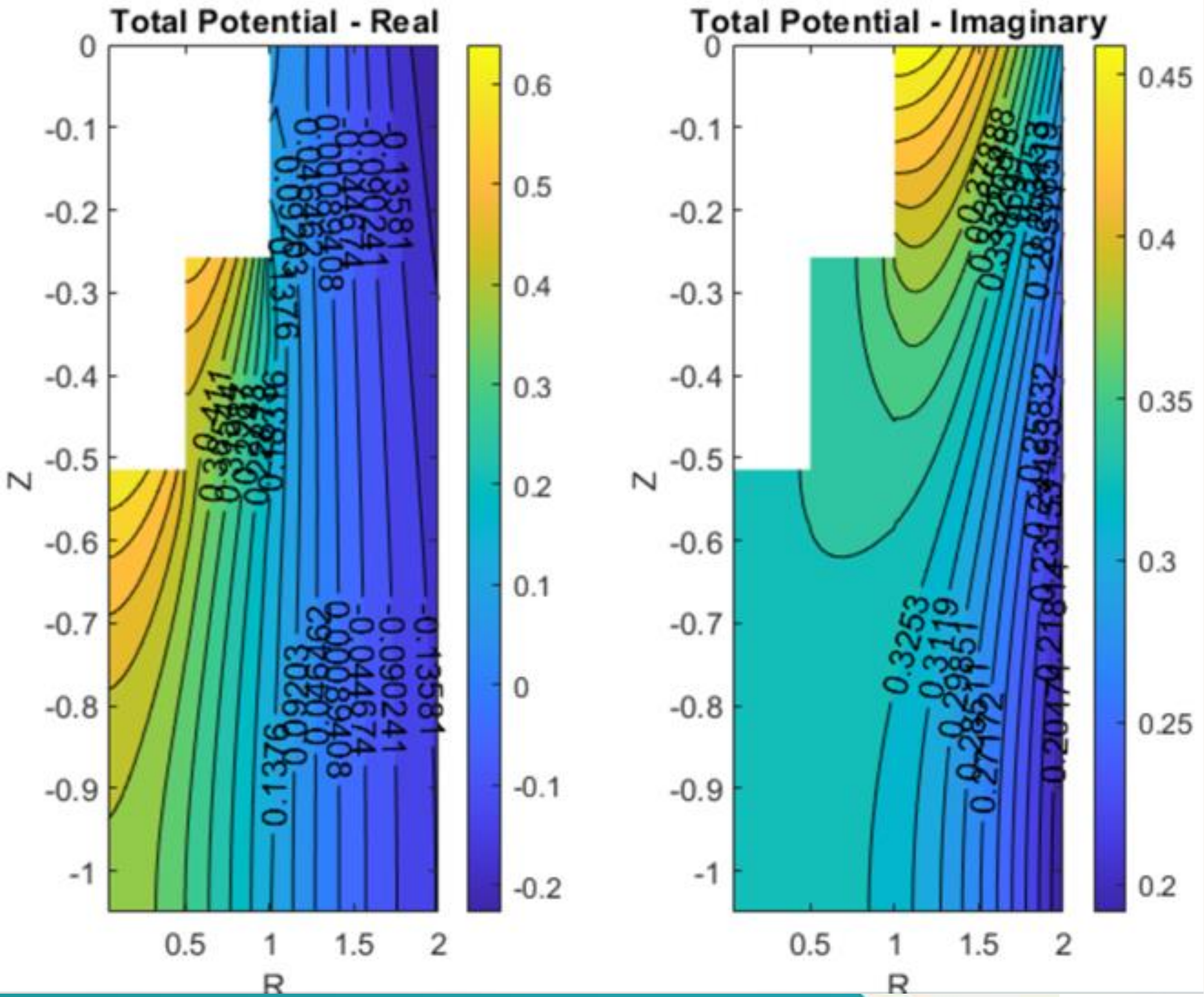
A matrix

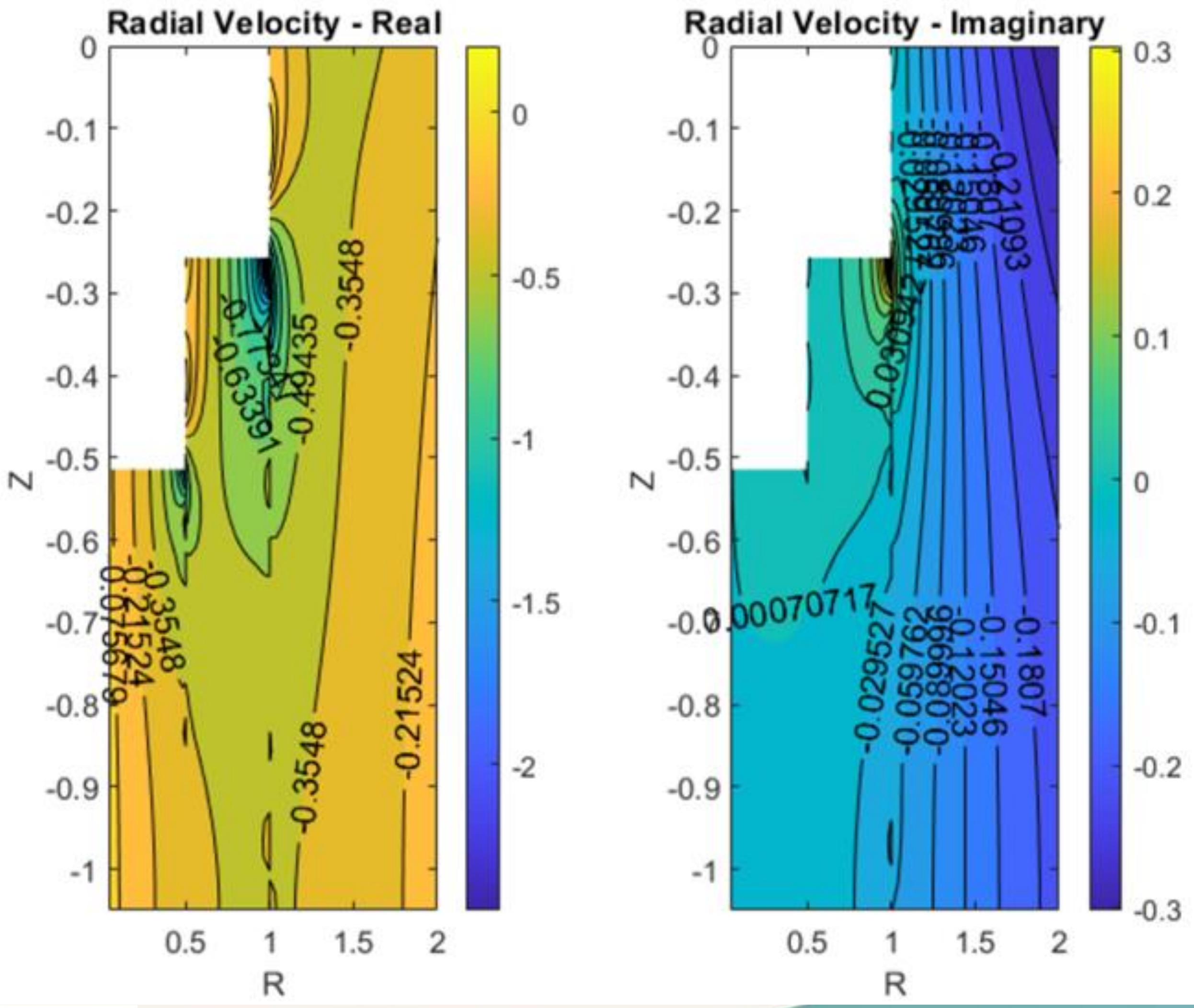
b vector

			$\overrightarrow{C_{1n}^{i1}}$	$\overrightarrow{C_{1m}^{i2}}$	$\overrightarrow{C_{2m}^{i2}}$	$\overrightarrow{B_k^{e}}$
Equation	r	size	N	M	M	K
$\phi^{i1} = \phi^{i2}$	a_1	N	$(h-d_1)\,\mathrm{diag}\left(\overrightarrow{R_{1n}^{i1}}\right)$	$-I_{nm} \odot 1_{N1}\overrightarrow{R_{1m}^{i2}}$	$-I_{nm} \odot 1_{N1}\overrightarrow{R_{2m}^{i2}}$	0_{NK}
$\phi^{i2} = \phi^{e}$	a_2	M	0_{MN}	$(h-d_2)\,\mathrm{diag}\left(\overrightarrow{R_{1m}^{i2}}\right)$	$(h-d_2)\,\mathrm{diag}\left(\overrightarrow{R_{2m}^{i2}}\right)$	$-I_{mk} \odot 1_{M1}\overrightarrow{\Lambda_k}$
$\frac{\partial}{\partial r}\phi^{i1} = \frac{\partial}{\partial r}\phi^{i2}$	a_1	M	$-I_{mn} \odot 1_{M1}\frac{\partial}{\partial r}\overrightarrow{R_{1n}^{i1}}$	$(h-d_2)\,\mathrm{diag}\left(\frac{\partial}{\partial r}\overrightarrow{R_{1m}^{i2}}\right)$	$(h-d_2)\,\mathrm{diag}\left(\frac{\partial}{\partial r}\overrightarrow{R_{2m}^{i2}}\right)$	0_{MK}
$\frac{\partial}{\partial r}\phi^{i2} = \frac{\partial}{\partial r}\phi^{e}$	a_2	K	0_{KN}	$-I_{km} \odot 1_{K1}\frac{\partial}{\partial r}\overrightarrow{R_{1m}^{i2}}$	$-I_{km} \odot 1_{K1}\frac{\partial}{\partial r}\overrightarrow{R_{2m}^{i2}}$	$h\,\mathrm{diag}\left(\frac{\partial}{\partial r}\overrightarrow{\Lambda_k}\right)$

N	$\int_{-h}^{-d_1} (\phi_p^{i2} - \phi_p^{i1})\overrightarrow{Z_n^{i1\,T}}\,dz$
M	$-\int_{-h}^{-d_2} \phi_p^{i2}\overrightarrow{Z_m^{i2T}}\,dz$
M	$\int_{-h}^{-d_1} \frac{\partial}{\partial r}\phi_p^{i1}\,\overrightarrow{Z_m^{i2T}}\,dz - \int_{-h}^{-d_2} \frac{\partial}{\partial r}\phi_p^{i2}\,\overrightarrow{Z_m^{i2T}}\,dz$
K	$\int_{-h}^{-d_2} \frac{\partial}{\partial r}\phi_p^{i2}\overrightarrow{Z_k^{eT}}\,dz$

Appendix: Potential and Velocity Fields





N=M=K=11